Summary of Methods for Second Order DE's

Consider the second order differential equation $y'' + a(t)y' + b(t)y = f(t)$. $\qquad (*)$

1. If the functions $a(t)$ and $b(t)$ are constants, it is always possible to find two linearly independent solutions of the corresponding undriven equation. A particular solution can then be found using variation of parameters. Thus, the complete solution can be found. If $f(t)$ is of the appropriate form, undetermined coefficients may be easier to use than variation of parameters to find a particular solution.
2. If the functions $a(t)$ and $b(t)$ are not constants, there is no general method for solving $(*)$ in terms of a finite number of elementary functions. [Solutions involving infinite series are possible but we don't cover that topic.] However, if one solution can be found for the corresponding undriven equation, then a second solution can be found using reduction of order, and the general solution can be obtained using variation of parameters. In fact, both of these calculations can be performed simultaneously, as is illustrated below. Thus, the problem of solving $(*)$ when $a(t)$ and $b(t)$ are not constants rests on the possibility of finding one solution to the corresponding undriven equation.

Example. Consider $ty'' + 2(1-t)y' + (t-2)y = 2e^t$. Observe that $y = e^t$ is a solution to the corresponding undriven equation. Set $y = e^t \cdot v(t)$. Then

$$[t(v'' + 2v' + v) + (2-2t)(v' + v) + (t-2)v]e^t = 2e^t$$

or

$$tv'' + 2v' = 2.$$

Then

$$v'(t) = -\frac{c_1}{t^2} + 1$$

So,

$$v(t) = \frac{c_1}{t} + t + c_2.$$

The general solution is then $y(t) = \left[\frac{c_1}{t} + t + c_2\right]e^t$.

Exercises. Find the general solution of the given equation, given one solution of the undriven differential equation.

1. $t^3y'' + ty' - y = 0$; $y = t$.
2. $2ty'' + (1-4t)y' + (2t-1)y = e^t$; $y = e^t$.
3. $(t^2+1)y'' - 2ty' + 2y = 6(t^2+1)^2$; $y = t$.