The Undetermined Coefficients Method for 2^{nd} Order Linear Differential Equations

Consider the following linear second order differential equation:

$$y'' + a(t)\,y' + b(t)\,y = f(t) \qquad (*)$$

In order to describe the method of undetermined coefficients, we first need a definition.

DEFINITION. A function is called a *UC function* if it is either

1. a function of one of the following types:
   (a) t^n, where n is a non-negative integer;
   (b) $e^{\alpha t}$, where α is a non-zero constant;
   (c) $\sin(\beta t + \gamma)$, where β and γ are constants, $\beta \neq 0$;
   (d) $\cos(\beta t + \gamma)$, where β and γ are constants, $\beta \neq 0$; or
2. a function defined as a finite product of functions listed in (1).

The method of undetermined coefficients will apply when $f(x)$ in $(*)$ is a finite linear combination of UC functions. Note that successive derivatives of a UC function are also UC functions or linear combinations of UC functions. This property of UC functions is what allows the method of undetermined coefficients to work.

DEFINITION. Let f be a UC function. The *UC set of f* consists of f itself together with all linear independent UC functions of which derivatives of f are constant multiples or linear combinations.

EXAMPLES

1. Let $f(t) = t^3$. Then f is a UC function, and the UC set of f is $\{t^3, t^2, t, 1\}$.
2. Let $f(t) = \sin 2t$. Then the UC set of f is $\{\sin 2t, \cos 2t\}$.
3. Let $f(t) = t^2 \cos t$. Then f is a UC function since it is a product of the functions t^2 and $\cos t$. The UC set of f is $\{t^2\cos t, t^2 \sin t, t\cos t, t\sin t, \cos t, \sin t\}$.

Now, in the differential equation $(*)$, suppose that $f(t)$ is a linear combination of UC functions u_i,

$$f(t) = a_1 u_1 + a_2 u_2 + \ldots + a_m u_m\,.$$

The method of undetermined coefficients prodeeds as follows.

1. Find a solution, y_u, (sometimes called a *complementary* solution) for the corresponding homogeneous (undriven) differential equation. Note that this is generally possible only if the coefficient functions $a(t)$ and $b(t)$ are constant functions. If they are not, we do not have a general procedure for finding the complementary solution.

2. For each of the UC functions
$$u_1, u_2, \ldots, u_m$$
of which f is a linear combination, form the respective UC sets, say
$$S_1, S_2, \ldots, S_m \; .$$
3. For each $i \neq j$, if $S_i \subset S_j$, delete S_i from the list.
4. For each remaining UC set which contains an element which is a solution of the homogeneous equation, multiply each member of that UC (and only that UC set) by the smallest integral power of t so that none of the members of that set solves the homogeneous equation.
5. Now form a linear combination of all the members of all the sets in step 4 using arbitrary constant coefficients (undetermined coefficients).
6. Determine the unknown coefficients by substituting the linear combination formed in the previous step into the original differential equation.

The procedure is illustrated in the following example.

EXAMPLE. Solve the following differential equation using the method of undetermined coefficients.

$$y'' - 3y' + 2y = 2t^2 + e^t + 2te^t + 4e^{3t}$$

Step 1: The corresponding homogeneous (undriven) equation has solution
$$y_c = c_1 e^t + c_2 e^{2t} \; .$$

Step 2: The function $f(t)$ on the right hand side of the original differential equation is a linear combinatin of the four UC functions
$$t^2, \; e^t, \; te^t, \text{ and } e^{3t} \; .$$
So we form the four UC sets of these functions to get $S_1 = \{t^2, \; t, \; 1\}$, $S_2 = \{e^t\}$, $S_3 = \{te^t, \; e^t\}$, and $S_4 = \{e^{3t}\}$.

Step 3: Since $S_2 \subset S_3$, we eliminate S_2 from further consideration.

Step 4: Note that S_3 includes e^t, which is a solution of the homogeneous (undriven) equation. So we multiply each member of S_3 by t to obtain $S_3' = \{t^2e^t, \; te^t\}$ which now has no members which are solutions of the homogeneous (undriven) equation.

Step 5: Forming linear combinations of the elements of S_1, S_3', and S_4, we obtain
$$y_d = At^2 + Bt + C + De^{3t} + Et^2e^t + Fte^t \; .$$

Step 6: Computing y_d' and y_d'' and substituting into the original driven ODE, one finds after some algebra that
$$y_d = t^2 + 3t + \frac{7}{2} + 2e^{3t} - t^2e^t - 3te^t \; .$$
Thus, the general solution is
$$y = y_u + y_d = c_1e^t + c_2e^{2t} + t^2 + 3t + \frac{7}{2} + 2e^{3t} - t^2e^t - 3te^t \; .$$