Variation of Parameters

Consider the following linear second order differential equation:

$$y'' + a(t)\,y' + b(t)\,y = f(t) \tag{1}$$

If

$$y_u(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2}$$

is the general solution to the corresponding undriven equation, the method of *variation of parameters* allows us to search for a particular solution to Equation (1) of the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \tag{3}$$

The method begins by assuming that we have a general solution $y_u(t)$ to the corresponding undriven equation as in Equation (2). This is no small assumption since we have only learned methods for doing this in very special cases (e.g., when $a(t)$ and $b(t)$ are constants). Nevertheless, we proceed assuming we can find a general solution to the undriven analogue of Equation (1) in the form of Equation (2). Then we can compute the first two derivatives of $y_p(t)$ and substitute them into the ODE (1). This would give us a single equation involving v_1 and v_2, so we have some latitude in imposing a second condition on those functions to simplify the work.

If we take $v_1'(t)y_1(t) + v_2'(t)y_2(t) = 0$ as the second condition, we will have two equations in the two unknown functions $v_1(t)$ and $v_2(t)$ as follows:

$$v_1'(t)y_1(t) + v_2'(t)y_2(t) = 0 \tag{4}$$

$$v_1'(t)y_1'(t) + v_2'(t)y_2'(t) = f(t) \tag{5}$$

Exercise 1: Verify that if Equation (3) is substituted into Equation (1) with the condition in Equation (4) imposed, then Equation (5) results.

We can use Cramer's Rule to solve the system given by Equations (4) and (5) to obtain

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix}}{W[y_1, y_2](t)} \tag{6}$$

$$v_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & f(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{\begin{vmatrix} y_2(t) & 0 \\ y_2'(t) & f(t) \end{vmatrix}}{W[y_1, y_2](t)} \tag{7}$$

where $W[y_1, y_2](t)$ denotes the Wronskian of $y_1(t)$ and $y_2(t)$.

Exercise 2: Verify this, i.e., compute the first two derivatives of y_p from Equation (3), substitute them into the ODE 1 and show Equation (5) follows once Equation (4) is imposed.

EXERCISE 3: Use the variation of parameters method to find a particular solution of the ODE

$$2y'' - 3y' + y = 10\cos t$$

The write the general solution of the ODE. [NOTE: This ODE is not in the form of Equation (1)]

EXERCISE 4: Find a particular solution and the general solution of

$$y'' + y = \sec t$$

EXERCISE 5: Consider the ODE

$$(1-t)y'' + ty' - y = 2(1-t)^2e^{-t}, \qquad t \in (1, \infty) \tag{8}$$

(a) Show that $\{y_1(t) = t, y_2(t) = e^t\}$ is a basic solution set for the undriven differential equation $(1-t)y'' + ty' - y = 0$ over the t-interval $(1, \infty)$.

(b) Write down the general solution of the undriven differential equation given in (a) that is valid over the t-interval $(1, \infty)$.

(c) Use variation of parameters to find a particluar solution of Equation (8). Then write down the general solution for this ODE.